ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Սահակյան Ալբերտ Խաչիկի

Գրաֆների միջակայքային ներկումներ նախապես տրված սահմանափակումներով

ՍԵՂՄԱԳԻՐ

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Albert Khachik Sahakyan

Interval colorings of graphs with predefined restrictions

SYNOPSIS

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Ատենախոսության թեման հաստատվել է Երևանի պետական համալսարանում:

Գիտական ղեկավար՝	ֆիզ.-մաթ. գիտ. դոկտոր Ռ. Ռ. Քամալյան
Պաշտոնական ընդդիմախոսներ՝	ֆիզ.-մաթ. գիտ. դոկտոր Ա. Վ. Կոնոնով
	ֆիզ.-մաթ. գիտ. թեկնածու Պ. Ա. Պետրոսյան
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Dissertation topic was approved at the Yerevan State University.

Scientific supervisor:	Doctor of Physico-Mathematical Sciences R. R. Kamalian
Official opponents:	Doctor of Physico-Mathematical Sciences A. V. Kononov
	Candidate of Physico-Mathematical Sciences P. A. Petrosyan
Leading organization:	Russian-Armenian University

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Scientific secretary of specialized council,
Doctor of Physico-Mathematical Sciences:  T. N. Harutyunyan

General characteristics of the work

The relevance of the topic

Graph coloring theory has an important position in discrete mathematics. The origins of graph coloring trace back to 1852 of the Four-Coloring Problem (FCP), which Cayley published in 1878. The formulation of the problem was: what is the least possible number of colors needed to fill in any map on the plane? Kempe presented the first "proof" of the FCP. Heawood stated that he had discovered an error in Kempe's proof. However, Heawood was able to prove that every map could be 5-colored. The FCP was one of the hottest topics in graph theory, and it had the attention of various mathematicians for a long time. In 1976, Appel and Haken announced a complete proof. The Four-Coloring Problem is a special case of the vertex-coloring problem since it requires finding the chromatic number of graphs that belong to a particular class of graphs.

In 1880 Tait considered coloring the edges of the graph. He proved that the FCP is equivalent to the problem of edge-coloring every planar 3-connected cubic graph with three colors. The planarity condition was added by Petersen twenty years after Tait's paper. The Edge-Coloring Problem (ECP) requires finding the chromatic index $\chi'(G)$ of a given graph G (the minimum number of colors needed to color the edges of G such that no two adjacent edges receive the same color).

In 1916 Kőnig proved that every bipartite graph can be Δ-edge-colored, where Δ is the maximum vertex degree. Later, Vizing proved that every simple graph can be edge-colored with $\Delta(G) + 1$ colors.

Compared with vertex coloring, the theory of edge-coloring has received less attention until recently. However, many studied areas, such as map coloring, matching theory, factorization theory, Latin squares, and scheduling theory, all have strong connections to the edge-coloring. The cornerstones in the theory of edge-coloring are the Shannon's and Vizing's bounds for the chromatic index in terms of the maximum degree and the maximum multiplicity, the Adjacency lemma for simple graphs as well as for multigraphs.

The vertex-coloring and edge-coloring problems for general graphs are NP-complete. Edge-coloring can be regarded as a special case of vertex-coloring since we can reduce it to the problem of coloring the vertices of the corresponding line graph.

Graph coloring appears in many places with seemingly no or little connection to coloring. A good example is the Erdiős-Stone-Simonovits theorem. Graph coloring deals with the fundamental problem of partitioning a set of objects into classes according to specific rules. Timetabling, sequencing, and scheduling problems, in their many forms, are basically of this nature.

In many practical applications finding a coloring is not enough because there are restrictions on the vertices or edges for the available colors. This has led to another crucial type of graph colorings called list coloring. In the list coloring problem, for each vertex $v \in V(G)$ a set $L(v)$ of allowed colors is provided and the color for each vertex should be from that list. The same problem for edge-coloring is called list edge-coloring. List coloring was first studied in the 1970s in independent papers by Vizing [1] and by Erdős, Rubin, and Taylor [2]. A graph is k-choosable if it has a proper list coloring, no matter how one assigns a list of k colors to each vertex. The list chromatic number $ch(G)$ of a graph G is the least number k such that G is k-choosable.

Usually, two problems are considered in this type of colorings:

1. Giving a polynomial algorithm that will find a coloring for any lists.

2. Find the list chromatic number for certain types of graphs.

Graph colorings with restrictions find more real-world applications from determining register assignments for computing processes to channel assignment in wireless networks. Another example is the chemical storage problem where radioactive substances might require special storage facilities. Thus in the corresponding graph, there is a list of colors (appropriate storage compartments) associated with each vertex (chemical). In an admissible coloring (assignment of compartments to chemicals), the color of a vertex must be chosen from its list.

List coloring problems are harder to solve than classical graph coloring problems. The list coloring problem is NP-complete for general perfect graphs, and is also NP-complete for many subclasses of perfect graphs, including split graphs, interval graphs, and bipartite graphs. In [6] a polynomial solution was provided for partial k-trees with the time complexity of $O(|V(G)|^{k+2})$. This variant received a considerable amount of attention that led to several fascinating conjectures and results and its study combines interesting combinatorial techniques with powerful algebraic and probabilistic ideas. Similar to list chromatic number, the list chromatic index is defined for list edge-coloring. The most researched problem here is the list coloring conjecture, which asserts that the list chromatic index is equal to the chromatic index for every graph. It has been proven for several types of graphs.

Later Kubale [7] considered a more general problem where each vertex or edge is being colored with an interval of integers with given lengths, and there can be forbidden colors for each vertex.

In the construction of timetabling without "gaps", a different kind of graph colorings were considered known as interval edge-coloring, first introduced by Asratian and Kamalian [4]. An edge-coloring of a graph G with consecutive integers $c_1, \ldots, c_t$ is called an interval t-coloring, if all colors are used, and the colors of edges incident to any vertex of G are distinct and form an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively. Interval edge-colorings have been intensively studied in different papers for different types of graphs. In [5] it was shown that every tree is from $\mathfrak{N}$. Lower and upper bounds on the number of colors in interval edge-colorings were provided in [9] and the bounds were improved in differe papers for different graphs: planar graphs, r-regular graphs with at least $2r + 2$ vertices, cycles, trees, complete bipartite graphs, n-dimensional cubes, complete graphs, Harary graphs, complete k-partite graphs.

It is known that not all bipartite graphs have interval edge-coloring. Mirumyan obtained the first such example in 1989, but it was not published. Later several papers were published on this topic. It was shown that the problem of finding an interval edge-coloring for bipartite graphs is NP-complete. Interval edge-colorings were extensively studied by many researchers.

In many applications of interval edge-colorings, some restrictions should be met. For example, when scheduling exams, we should also take into account the availability of the teachers. Similar to list coloring problems, we need to find an interval edge-coloring that satisfies the given lists of available colors for each edge. In practice, scheduling problems like school timetables, where colors represent time slots, particular participants, or resources are only available within certain time intervals. Interval colorings with restrictions solve these issues, but since they are generalizations of NP-hard problems, they are also NP-hard. When considering interval edge-colorings with restrictions, we need to satisfy the classical graph coloring conditions, interval edge-coloring conditions, and the restrictions on the colors. Hence solving these problems is significantly harder than finding interval edge-colorings, and because of it, they did not get enough attention. Even in the case of trees where there are restrictions on the edges for the available colors the problem of finding interval edge-coloring that meets those restrictions were not solved until recently. The restrictions can be on the edges, on the vertices, or on the spectrums. For some graphs, the problem with restrictions on spectrums can have a polynomial solution, but the problem with restrictions on the edges might not.

This work is dedicated to interval edge-colorings with restrictions but also contains new results on different types of colorings that are similar in nature and can be useful in understanding the structure and the complexity of interval edge-colorings with restrictions.

The main purpose of the work and the considered problems

In this work, we considered interval edge-colorings of graphs with restrictions. The restrictions can be on the edges, on the vertices, or on the spectrums. Polynomial solutions were provided for different classes of graphs and for different types of restrictions. For some classes of graphs, we proved that the problem is NP-complete. We also provided new results in interval edge-colorings, list colorings, and interval vertex-colorings for some classes of graphs. The main purpose of the work is to research graph colorings with given restrictions.

The main results are summarized in Table 1. For each problem and each class of graph, its appropriate cell in the table represents the result. If the problem has a polynomial solution, the cell has the following format: *P[reference]*. If the problem was proved to be NP-complete, it has the following format: *NPC[reference]*. The cells with the highlighted background represent the results we obtained and contain a reference to the section.

The following classes of graphs were considered: trees, cyclic trees (each vertex is inside one cycle at most), cycles with chords (Definition 2.2.1), bipartite cactus graphs, cactus graphs (each cycle is inside one cycle at most), complete bipartite graphs, complete graphs, block graphs (each block is a clique).

The following problems were considered.

E1: For a given graph, find any interval edge-coloring.

E2: For a given graph, find an interval edge-coloring with minimal number of colors.

E3: For a given graph, find an interval edge-coloring for given restrictions on spectrums.

E4: For a given graph, find an interval edge-coloring for given restrictions on vertices.

E5: For a given graph, find an interval edge-coloring for given restrictions on edges.

L: For a given graph, find a vertex coloring for given restrictions on vertices.

I: For a given graph, find an interval vertex-coloring for given restrictions on vertices.

Graph \ Problem	E1	E2	E3	E4	E5	L	I
Tree	P[5]	P[5]	P(1.2)	P(1.3)	P(1.3)	P[10]	P[7]
Cyclic Tree	P(2.1)	P(2.1)	P(2.1)	P(2.1)	P(2.1)	P(2.4)	P(2.4)
Cycle with chords	P(2.2)	P(2.2)	P(2.2)	P(2.2)	P(2.2)	P(2.4)	P(2.4)
Bipartite Cactus	P[8]	P[8]	P(2.3)	NPC(2.5)	NPC(2.5)	P(2.4)	P(2.4)
Cactus	-	-	P(2.3)	NPC(2.5)	NPC(2.5)	P(2.4)	P(2.4)
$K_{n,m}$	P[5]	P[5]	P(3.4)[1]	P(3.4)[1]	NPC(3.4)	NPC(3.3)	NPC(3.3)
K_n	P[9]	P[9]	-	-	NPC(3.5)	P[10]	NPC[7]
Block Graph	P(3.1)[2]	P(3.1)[2]	-	NPC(2.5)	NPC(3.5)	P(3.2)	NPC[7]

Table 1: The graph-class/problem solvability map. Each cell gives a reference to the paper or to the section in this work and information about whether the problem has a polynomial solution or was proved to be NP-complete. The highlighted cells represent the results obtained by us.

Research objects

The main objects of the research are different classes of graphs, interval edge-colorings of graphs, list colorings of graphs, interval vertex-colorings of graphs. We are interested in the graph classes for which there is a polynomial algorithm for finding interval edge-coloring for any restrictions on the edges, vertices, or spectrums. We are also interested in the graph classes for which the problem is NP-complete. In most cases, we are interested in graph coloring problems with some restrictions.

Research methods

The research was done using methods of discrete mathematics, graph theory, discrete optimization, computer science, dynamic programming, algorithms, and data structures. Some algorithms were also validated with implementing computer programs.

Scientific innovation

This is a first-of-its-kind work dedicated to interval edge-colorings with given restrictions of colors. Interval edge-colorings and list edge-colorings are important topics of graph theory.

1. The problem was solved for the subclass of $K_{n,m}$ where n, m are coprime.
2. The problem was solved for even block graphs (the number of vertices in each block is even).

Interval edge-colorings with restrictions is the combination of these two topics. The solution was previously unknown even for the simple classes of graphs like trees. When the restrictions are on the edges, we give polynomial solutions for trees, cyclic trees, and cycles with chords, and prove that the problem is NP-complete for cactus graphs, complete bipartite graphs, and complete graphs. We give a solution for a subclass of complete bipartite graphs when the restrictions are on the vertices. When the restrictions are on the spectrums, we give a polynomial solution for cactus graphs.

Practical applications of the obtained results

The results of this work give new techniques and theorems that open a new door for investigating graph colorings with restrictions. Most importantly, in real-world applications of graph colorings and interval edge-colorings, multiple restrictions should be taken into account. In particular, interval edge-colorings are being used in constructing compact schedulings, constructing school timetables, constructing sports schedules, solving chemical storage problems, assignments for computing processes, channel assignments in wireless networks, optimal distribution of memory for computer programs, mathematical models with continuous processes. But in practice, there are restrictions on the available resources, and we aim to solve those issues.

The main provisions of the defense

The following main provisions are presented for defense

1. Polynomial algorithms of finding an interval edge-coloring with given restrictions on spectrums for trees and cactus graphs.
2. Polynomial algorithms for finding an interval edge-coloring with given restrictions on edges for trees, cyclic trees, and cycles with chords.
3. A polynomial algorithm for interval vertex-coloring with given restrictions on vertices for cactus graphs.
4. A polynomial algorithm for list coloring of block graphs.
5. A polynomial algorithm for finding an interval edge-coloring with given restrictions on vertices for a subclass of complete bipartite graphs.
6. Interval edge-coloring of even block graphs and estimations for the minimal number of required colors.
7. NP-completeness of list coloring for complete bipartite graphs.
8. NP-completeness of interval edge-coloring with restrictions on edges for bipartite cactus graphs, complete, and complete bipartite graphs.
9. NP-completeness of interval edge-coloring with restrictions on vertices for bipartite cactus graphs, and block graphs.

Approbation and testing of the obtained results

The main results were presented and discussed in multiple seminars at Yerevan State University, Russian-Armenian University, Institute for Informatics and Automation Problems of National Academy of Sciences of the Republic of Armenia, and international conferences: Mathematics and IT: Research and Education (MITRE 2021, Moldova), 7th International Scientific Conference dedicated to the 30th anniversary of the Independence of the Republic of Armenia.

Publications

In the scope of the research, 8 scientific papers are published in different journals.

The size and the structure of the work

The size of the dissertation is 102 pages. It contains an introduction, three chapters, a conclusion, and references. The first chapter is dedicated to general results and trees. The second chapter is dedicated to graphs with a special structure of cycles (cyclic trees, cycles with chords, and cactus graphs). The third chapter is dedicated to different types of complete graphs (complete graphs, complete bipartite graphs, and block graphs). Reading the section 1.1 about general results is required for all the chapters. Other than that, all three chapters can be read independently. The work contains 1 table and 38 images.

The main results of the thesis

All graphs considered in this work are undirected (unless explicitly said), finite, and have no loops or multiple edges. For an undirected graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$. The maximum degree of vertices in G is denoted by $\Delta(G)$. Our definitions follow [11].

A tree is a connected graph that has no cycles. A cactus is a connected graph where any two simple cycles have at most one vertex in common. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U_1 and U_2 such that every edge connects a vertex in U_1 to a vertex in U_2. A complete bipartite graph is a bipartite graph such that two vertices are adjacent, if and only if they are in different partite sets. When the sets have sizes n and m, the complete bipartite graph is denoted by $K_{n,m}$. When we say given a complete bipartite graph $K_{n,m}$, we will assume that $n \geq m$. A complete graph is a graph in which every pair of distinct vertices is connected by an edge. The complete graph having n vertices is denoted by K_n. A vertex is a cut vertex in a connected graph if its removal makes the graph disconnected. A block of a graph G is a maximal connected subgraph of G that has no cut vertex.

The greatest common divisor of two positive integers a, b is denoted by $\sigma(a,b)$. The set of integers $\{a, a+1, \dots, b\}$, $a \leq b$, is denoted by $[a,b]$. For a finite set S let 2^S be the set of all the subsets of the set S. For a finite set of numbers, let $max(S)$ be the maximal element in that set and $min(S)$ be the minimal element in that set. The set of positive integers is denoted by $\mathbb{N}$. For two positive integers a and n, the remainder of a when divided by n is denoted by $a \bmod n$.

Let I_k be the set $[1, k]$ of integers. We will denote by $\tau(I_k)$ the set of all the elements from 2^{I_k} that are an interval of integers. More formally $\tau(I_k) = \{s : s \in 2^{I_k}, s \text{ is a non empty interval of integers}\}$.

A vertex coloring is an assignment of colors to each vertex of a graph such that every two adjacent vertices have different colors. The minimum number of colors with which the vertices of a graph G may be colored is called the chromatic number, denoted $\chi(G)$.

For a graph G and a function $L : V(G) \to 2^{I_k}$, a list coloring $\beta : V(G) \to I_k$ is a coloring of the graph vertices with integers from I_k such that for every vertex v, $\beta(v) \in L(v)$ and every two adjacent vertices have different colors. List coloring is a generalization of vertex coloring since if we take $L(v) = I_k$ for all $v \in V(G)$ then finding the minimal k for which the list coloring has a solution is equivalent to finding $\chi(G)$.

An interval vertex-k-coloring of a graph G is a function $\gamma : V(G) \to \tau(I_k)$ such that $\gamma(u) \cap \gamma(v) = \emptyset$ for all the edges $(u, v) \in E(G)$. In other words, we are coloring the vertices with intervals of integers such that for every two adjacent vertices, their respective intervals do not intersect.

An edge-coloring of a graph G is an assignment of colors to the edges of the graph so that no two adjacent edges have the same color. We are going to consider only edge-colorings with positive integers. For an edge-coloring $\alpha : E(G) \to \mathbb{N}$ and a vertex v, the set of all the colors of the incident edges of v is called the spectrum of that vertex in α. The spectrum of the vertex v is denoted by $S_\alpha(v)$. The smallest and the largest numbers in $S_\alpha(v)$ are denoted by $\underline{S_\alpha}(v)$ and $\overline{S_\alpha}(v)$, respectively.

An interval edge-coloring of a graph G is an edge-coloring with positive integers such that for each vertex v the colors of edges incident to v form an interval of integers. An interval edge-coloring of a graph G with colors $1, \ldots, t$ is an interval t-coloring if all colors are used. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. This means that an interval t-coloring is a function $\alpha : E(G) \to [1, t]$ such that for each color from 1 to t, there is an edge with that color, and for each vertex v all the edges incident to v have different colors forming an interval of integers.

We consider the following seven problems for different classes of graphs. In all the problems, we assume that the graph is connected.

Problem 1. ***(E1): Interval edge-coloring***
Given a graph G. Find any interval edge-coloring. More formally, find an edge-coloring $\alpha : E(G) \to \mathbb{N}$ such that for each vertex v all the edges incident to v have different colors forming an interval of integers.

Problem 2. ***(E2): Interval edge-coloring with minimal colors***
Given a graph G. Find the minimal t for which there is an interval t-coloring. More formally, find an edge-coloring $\alpha : E(G) \to [1, t]$ (with minimal t) such that for each color from 1 to t, there is an edge with that color, and for each vertex v all the edges incident to v have different colors forming an interval of integers.

Problem 3. ***(E3): Interval edge-coloring with restrictions on spectrums***
Given a graph G, and for some k, restrictions on the spectrums $S : V(G) \to \tau(I_k)$ with $|S(v)| = d_G(v)$ for all $v \in V(G)$. Find an interval edge-coloring $\alpha : E(G) \to I_k$ such that $S_\alpha(v) = S(v)$ for all $v \in V(G)$.

Problem 4. ***(E4): Interval edge-coloring with restrictions on vertices***
Given a graph G, and for some k, restrictions on the vertices $L : V(G) \to 2^{I_k}$. Find an interval edge-coloring $\alpha : E(G) \to I_k$ such that $S_\alpha(v) \subseteq L(v)$ for all $v \in V(G)$.

Problem 5. ***(E5): Interval edge-coloring with restrictions on edges***
Given a graph G, and for some k, restrictions on the edges $R : E(G) \to 2^{I_k}$. Find an interval edge-coloring $\alpha : E(G) \to I_k$ such that $\alpha(e) \in R(e)$ for all $e \in E(G)$.

Problem 6. ***(L): List coloring***
Given a graph G, and for some k, restrictions on the vertices $L : V(G) \to 2^{I_k}$. Find a vertex coloring $\beta : V(G) \to I_k$ such that $\beta(v) \in L(v)$ for all $v \in V(G)$.

Problem 7. ***(I): Interval vertex-coloring***
Given a graph G, and for some k, functions $l : V(G) \to I_k$ and $S : V(G) \to 2^{I_k}$. Find an interval vertex-k-coloring $\gamma : V(G) \to \tau(I_k)$ such that $|\gamma(v)| = l(v)$, $\gamma(v) \subseteq S(v)$ for every vertex $v \in V(G)$, and $\gamma(u) \cap \gamma(v) = \emptyset$ for every edge $(u, v) \in E(G)$.

We consider these problems for the following classes of graphs: trees, cyclic trees, cycles with chords, bipartite cactus graphs, cactus graphs, complete bipartite graphs, complete graphs, block graphs. For each problem and each class of graph, we either provide a polynomial solution or show that the problem is NP-complete. For some problems and classes of graphs, we solve the problem for a subclass of that class of graphs.

The first chapter is dedicated to interval edge-colorings with restrictions in the case for trees. The section 1.1 provides general results for the considered problems and are being used from the other sections.

We prove a lemma for the connection between the considered problems.

Lemma 1.1.3. *The relationships between the complexities of the seven problems are the following.*

- *Solving the Problem 5 also solves the Problem 4.*
- *Solving the Problem 4 also solves the Problem 3.*
- *Solving the Problem 4 also solves the Problem 2.*
- *Solving the Problem 2 also solves the Problem 1.*
- *Solving the Problem 7 also solves the Problem 6.*

Then we prove a theorem that allows us reduce the problem of interval t-coloring to the problem of interval edge-coloring.

Theorem 1.1.4. *If we can detect in polynomial time for any restrictions R on the edges whether it is possible to find an interval edge-coloring that meets the restrictions R, then:*

(a) We can detect in polynomial time whether there is an interval edge-coloring that satisfies the restrictions R and uses the color 1.

(b) We can detect in polynomial time whether there is an interval edge-coloring that satisfies the restrictions R, uses the color t and all the colors are less than or equal to t.

(c) *We can detect in polynomial time whether there is an interval t-coloring that satisfies the restrictions R for any given integer t.*

Then we provide general results for edge-colorings with given restrictions on spectrums.

Lemma 1.1.6. *Let G be a connected graph with $|V(G)| \geq 2$ and let $r \in V(G)$ be an arbitrary vertex. Given spectrum restrictions $S(v)$ with $|S(v)| = d_G(v)$ for $v \in V(G) \setminus \{r\}$. For all edge-colorings α that satisfy the restrictions S for the vertices $V(G) \setminus \{r\}$ the spectrum $S_\alpha(r)$ is the same.*

Theorem 1.1.7. *Given a graph G with spectrums $|S(v)| = d_G(v)$ for $v \in V(G)$. For each block B of G, we can uniquely identify the spectrums in that block or identify that there is no edge-coloring. Hence if we can find an edge-coloring for each block, we can find an edge-coloring for the graph.*

After that, we prove that the considered decision problems are from the class NP.

Then we show that the problem of finding an interval edge-coloring with given restrictions on the spectrums is NP-complete.

Theorem 1.1.13. *The problem of finding an interval edge-coloring in cubic graphs that satisfies the given spectrum restrictions is NP-complete.*

Corollary 1.1.14. *The Problem 3 of finding an interval edge-coloring for given spectrum restrictions is NP-complete for general graphs.*

In the section 1.2 we provide a polynomial algorithm for finding an interval edge-coloring for given restrictions on the spectrums in the case of trees.

Theorem 1.2.1. *The Problem 3 of finding an interval edge-coloring of trees for given restrictions on spectrums can be solved in $O(N)$, where $N = |V(G)|$.*

In the section 1.3 we provide a polynomial algorithm for finding an interval edge-coloring for given restrictions on the edges in the case of trees.

Theorem 1.3.1. *The Problem 5 of finding an interval edge-coloring of trees for given restrictions on edges can be solved in $O(N^5)$, where $N = |V(G)|$.*

Corollary 1.3.2. *The Problem 4 of finding an interval edge-coloring of trees for given restrictions on vertices can be solved in polynomial time.*

In the second chapter, we consider the problems for graphs with a special structure of cycles (cactus graphs, cyclic trees, cycles with chords). Cyclic trees are similar to cactus graphs, but the main difference is that in the case of cactus graphs, a vertex can belong to multiple cycles, while in the case of cyclic trees, every vertex can belong to at most one cycle.

In the section 2.1 we provide a polynomial algorithm for finding an interval edge-coloring for given restrictions on the edges in the case of cyclic trees.

Theorem 2.1.1. *The Problem 5 of finding an interval edge-coloring of cyclic trees for given restrictions on edges can be solved in $O(N^6)$, where $N = |V(G)|$.*

Corollary 2.1.2. *The Problems 1, 2, 3, 4 can be solved in polynomial time for cyclic trees.*

In the section 2.1 we provide a polynomial algorithm for finding an interval edge-coloring for given restrictions on the edges in the case of cycles with chords.

Let G be a connected graph with N vertices. We say that G is labeled if the labels of the vertices are $1, \ldots, N$.

Definition 2.2.1. *Given a labeled graph G with N vertices. The graph G is called cycle with chords if:*

- $N \geq 3$.
- $(1,2) \in E(G), (2,3) \in E(G), \ldots, (N-1,N) \in E(G), (N,1) \in E(G)$. *These edges are called cycle edges.*
- *All the edges that are not cycle edges are called chords. For any two distinct chords $(u_1, v_1) \in E(G)$ ($u_1 < v_1$) and $(u_2, v_2) \in E(G)$ ($u_2 < v_2$), either $v_1 \leq u_2$ or $v_2 \leq u_1$.*

Theorem 2.2.2. *The Problem 5 of finding an interval edge-coloring of cycles with chords for given restrictions on edges can be solved in $O(N^4)$, where $N = |V(G)|$.*

Corollary 2.2.3. *The Problems 1, 2, 3, 4 can be solved in polynomial time for cycles with chords.*

In the section 2.3 we provide a polynomial algorithm for finding an interval edge-coloring for given restrictions on the spectrums in the case of cactus graphs.

Theorem 2.3.5. *The Problem 3 of finding an interval edge-coloring of cactus graphs for given restrictions on spectrums can be solved in $O(N^2)$, where $N = |V(G)|$.*

In the section 2.4 we provide a polynomial algorithm for finding an interval vertex-coloring for given restrictions on the vertices in the case of cactus graphs.

Theorem 2.4.1. *The Problem 7 of finding an interval vertex-coloring of cactus graphs can be solved in $O(N \cdot k^3)$, where $N = |V(G)|$ and k is the maximal allowed color.*

Corollary 2.4.2. *The Problem 6 of finding a list coloring of cactus graphs can be solved in polynomial time.*

Corollary 2.4.3. *The Problem 7 of finding an interval vertex-coloring for cycles with chords can be solved in polynomial time.*

In the section 2.5 we prove that the problems of finding an interval edge-coloring for given restrictions on the edges or vertices are NP-complete in the case of cactus graphs and bipartite cactus graphs.

Definition 2.5.1. *Let G be a cactus graph. If all the cycles have the length k, we say that G is a k-cycle cactus. The set of all k-cycle cactus graphs is denoted by C_k.*

Theorem 2.5.3. *The Problem 5 of finding an interval edge-coloring α in a 4-cycle cactus graph G ($G \in C_4$ and $t = |E(G)|$) that meets the edge restrictions $R : E(G) \to 2^{I_t}$ is NP-complete.*

Corollary 2.5.4. *The list edge-coloring problem is NP-complete for 4-cycle cactus graphs.*

Corollary 2.5.5. *The list edge-coloring problem is NP-complete for cactus graphs.*

Theorem 2.5.6. *The Problem 5 of finding an interval edge-coloring of cactus graphs that satisfies the given edge restrictions is NP-complete.*

Theorem 2.5.7. *The Problem 5 of finding an interval edge-coloring of bipartite cactus graphs that satisfies the given edge restrictions is NP-complete.*

Theorem 2.5.10. *The Problem 4 of finding an interval edge-coloring α in a 3-cycle cactus graph G ($t = |E(G)|$) that meets the vertex restrictions $L : V(G) \to 2^{I_t}$ is NP-complete.*

Theorem 2.5.11. *The Problem 4 of finding an interval edge-coloring α in a cactus graph that meets the vertex restrictions is NP-complete.*

Corollary 2.5.12. *The problem of finding an edge-coloring α in a cactus graph that meets the vertex restrictions is NP-complete.*

Theorem 2.5.13. *The Problem 4 of finding an interval edge-coloring α in a block graph that meets the vertex restrictions is NP-complete.*

Theorem 2.5.15. *The Problem 4 of finding an interval edge-coloring α in a 4-cycle cactus graph G ($t = |E(G)|$) that meets the vertex restrictions $L : V(G) \to 2^{I_t}$ is NP-complete.*

Corollary 2.5.16. *The problem of finding an edge-coloring that meets the given vertex restrictions is NP-complete for 4-cycle cactus graphs.*

Theorem 2.5.17. *The Problem 4 of finding an interval edge-coloring that meets the given vertex restrictions is NP-complete for bipartite cactus graphs.*

In the third chapter, we consider the problems for complete, complete bipartite, and block graphs.

A block graph is an even block graph if each block has an even number of vertices. In the section 3.1 we provide a polynomial algorithm for finding an interval edge-coloring in the case of even block graphs.

Theorem 3.1.2. *Every even block graph G has an interval edge-coloring. Furthermore $w(G) \leq 2 \cdot (\Delta(G) - 1)$ when $\Delta(G) \geq 2$.*

We also show that the inequality can not be improved.

Theorem 3.1.3. *For every even integer m there exists an even block graph G with $\Delta(G) = m$ for which $w(G) = 2 \cdot (\Delta(G) - 1)$.*

In the section 3.2 we provide a polynomial algorithm for finding a list coloring in the case of block graphs.

Theorem 3.2.1. *The Problem 6 of finding a list coloring of a block graph G can be solved in $O(N^2 \cdot k^2)$, where $N = |V(G)|$, and k is the maximal allowed color.*

In the section 3.3 we prove the the problem of finding finding a list coloring is NP-complete in the case of complete bipartite graphs.

Theorem 3.3.1. *The Problem 6 of finding a list coloring of complete bipartite graphs where lists can have at most three colors is NP-complete.*

In the section 3.3 we consider finding an interval edge-colorings of complete bipartite graphs for given restrictions on the edges or vertices.

We first provide a polynomial algorithm when the restrictions are on the vertices and $\sigma(n, m) = 1$.

Theorem 3.4.1. *The Problem 4 of finding an interval edge-coloring for given vertex restrictions can be solved in $O(k \cdot n^3)$ in the case of complete bipartite graph $K_{n,m}$ with $\sigma(n, m) = 1$, where k is the maximal allowed color.*

Corollary 3.4.2. *The Problem 3 of finding an interval edge-coloring for given spectrum restrictions can be solved in polynomial time in the case of complete bipartite graph $K_{n,m}$ with $\sigma(n, m) = 1$, where k is the maximal allowed color.*

Then we prove the problem of finding an interval edge-coloring for given restrictions on edges is NP-complete in the case of complete bipartite graphs.

Theorem 3.4.3. *The Problem of finding an edge-coloring for the graph $K_{n,n}$ with given restrictions $R : E(K_{n,n}) \to 2^{I_n}$ is NP-complete.*

Theorem 3.4.4. *The Problem 5 of finding an interval edge-coloring for a complete bipartite graph $K_{n,m}$ with given restrictions $R : E(K_{n,m}) \to 2^{I_{n+m-1}}$ is NP-complete.*

In the section 3.3 we prove that the problem of finding an interval edge-coloring for given restrictions on the edges is NP-complete in the case of complete graphs.

Lemma 3.5.2. *The Problem of finding an edge-coloring of a complete graph K_n with given restrictions $R : E(K_n) \to 2^{I_{n-1}}$ is NP-complete.*

Theorem 3.5.3. *Let $t = |E(K_{2\cdot m})|$. The Problem 5 of finding an interval edge-coloring of a complete graph $K_{2\cdot m}$ with given restrictions $R : E(K_{2\cdot m}) \to 2^{I_t}$ is NP-complete.*

Corollary 3.5.4. *Let G be a block graph with $t = |E(G)|$. The Problem 5 of finding an interval edge-coloring of the block graph G with given restrictions $R : E(G) \to 2^{I_t}$ on the edges is NP-complete.*

Finally, for the problem of finding an edge-coloring that satisfies the restrictions on the spectrums, we show that solving it for complete graphs will also solve it for block graphs.

Theorem 3.5.5. *The Problem of finding an edge-coloring for block graphs with given spectrums can be polynomially reduced to the Problem of finding edge-colorings of complete graphs with given spectrums.*

Conclusion

Graph colorings with restrictions have many real-world applications and studying such problems for different classes of graphs can be used for these applications and can also strengthen the theoretical aspect of it.

In this work, we considered seven problems for some classes of graphs.

- Problem 1: For a given graph, find any interval edge-coloring.
- Problem 2: For a given graph, find an interval edge-coloring with minimal number of colors.
- Problem 3: For a given graph, find an interval edge-coloring for given restrictions on spectrums.
- Problem 4: For a given graph, find an interval edge-coloring for given restrictions on vertices.

- Problem 5: For a given graph, find an interval edge-coloring for given restrictions on edges.
- Problem 6: For a given graph, find a vertex coloring (list coloring) for given restrictions on vertices.
- Problem 7: For a given graph, find an interval vertex-coloring for given restrictions on vertices.

The considered classes of graphs were: Trees, Cyclic trees (each vertex belongs to at most one cycle), Cycles with chords (Definition 2.2.1), Bipartite Cactus graphs, Cactus graphs, Complete bipartite graphs $K_{n,m}$, Complete graphs K_n, Block graphs. Table 1 is a great summary of what was done before and what was done by us. The Problem 1 is easier than the Problem 2, the Problem 2 and the Problem 3 are easier than the Problem 4, the Problem 4 is easier than the Problem 5, the Problem 6 is easier than the Problem 7. Consequently, if we solve the harder problem in polynomial time, then we also solve the easier problem in polynomial time too. Similarly, if we prove that the easier problem is NP-complete, then the harder problem is NP-complete too, since all the problems are from the class NP.

We provided polynomial algorithms of the Problem 5 for trees, cyclic trees, and cycles with chords, which solved the Problems 1, 2, 3, 4 for these classes of graphs.

We provided a polynomial algorithm for the Problem 7 for cactus graphs, which solved the Problems 6 and 7 for bipartite cactus graphs, cyclic trees, and trees. We proved that the Problems 4 and 5 are NP-complete for bipartite cactus graphs, which gives an understanding of the boundary of the graph classes when the polynomial solvability becomes NP-complete. The proofs were also correct for edge-coloring, which means finding an edge-coloring with given restrictions on edges or vertices is NP-complete for cactus and bipartite cactus graphs. Furthermore, we proved these results for 3-cycle and 4-cycle cactus graphs. We provided a polynomial solution for the Problem 3 for cactus graphs. This means that when we switch from spectrum restrictions to restrictions on vertices, the problem becomes NP-complete for cactus graphs.

We proved that the Problem 5 is NP-complete for complete and complete bipartite graphs. The proofs were valid for edge-coloring too, which means the list edge-coloring problem is NP-complete for complete and complete bipartite graphs. We proved that the list coloring problem (Problem 6) is also NP-complete for complete bipartite graphs. We provided a polynomial algorithm for the Problem 4 in the case of $K_{n,m}$ with $\sigma(n, m) = 1$.

We showed that for even block graphs (each block has an even number of vertices) $w(G) \leq 2 \cdot (\Delta(G) - 1)$ and we constructed an example for which $w(G) = 2 \cdot (\Delta(G) - 1)$ which means we can not improve the inequality $w(G) \leq 2 \cdot (\Delta(G) - 1)$ for the class of even block graphs. We proved that the Problem 5 is NP-complete for block graphs since it is NP-complete for complete graphs. We also showed that the Problem 4 is NP-complete for block graphs since the cactus example that was used for proving NP-completeness in the case of cactus graphs was also a block graph. In the case of block graphs, we provided a polynomial algorithm for the Problem 6. We also showed that the Problem 3 for block graphs can be reduced to the problem 3 for complete graphs.

There were still cases when we neither provided a polynomial solution of a problem nor proved its NP-completeness (such as the Problem 1 for cactus graphs, the Problem 3 for complete graphs).

List of author's publications

1. Sahakyan A. K., Kamalian R. R. INTERVAL EDGE-COLORINGS OF TREES WITH RESTRICTIONS ON THE EDGES. *Proceedings of the YSU A: Physical and Mathematical Sciences*, 2021, **55(2 (255))**, 113-122.
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3. Sahakyan A. K. EDGE COLORING OF CACTUS GRAPHS WITH GIVEN SPECTRUMS. *International Academy Journal Web of Scholar*, 2021, **(2 (52))**, 25-28.
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4. Sahakyan A. K. List Coloring of Block Graphs and Complete Bipartite Graphs. *World Science*, 2021, **8(69)**, 36-43.
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5. Sahakyan A., Muradyan L. Interval edge-coloring of even block graphs. *Norwegian Journal of Development of the International Science*, 2021, **(70-1)**, 26-30.
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6. Sahakyan A. INTERVAL EDGE-COLORING WITH RESTRICTIONS ON EDGES FOR GRAPHS WITH SPECIAL STRUCTURES OF CYCLES. *Norwegian Journal of Development of the International Science*, 2021, **(71)**, 13-19.
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Ամփոփում

Գրաֆների ներկումները՝ նախապես տրված սահմանափակումներով, ունեն բազմաթիվ կիրառություններ: Տվյալ խնդիրների դիտարկումը տարբեր տեսակի գրաֆների դասերի համար օգտակար է բազմաթիվ կիրառությունների դեպքում և կարող է նաև ունենալ որոշակի տեսական արժեք:

Աշխատանքում դիտարկված են յոթ խնդիրներ, գրաֆների որոշ դասերի համար:

- Խնդիր 1: Տրված գրաֆի համար գտնել միջակայքային կողային ներկում:
- Խնդիր 2: Տրված գրաֆի համար գտնել միջակայքային կողային ներկում նվազագույն քանակի գույներով:
- Խնդիր 3: Տրված գրաֆի համար գտնել միջակայքային կողային ներկում, երբ սպեկտրների վրա կան սահմանափակումներ:
- Խնդիր 4: Տրված գրաֆի համար գտնել միջակայքային կողային ներկում, երբ գագաթների վրա կան սահմանափակումներ:
- Խնդիր 5: Տրված գրաֆի համար գտնել միջակայքային կողային ներկում, երբ կողերի վրա կան սահմանափակումներ:
- Խնդիր 6: Տրված գրաֆի համար գտնել գագաթային ներկում, երբ գագաթների վրա կան սահմանափակումներ:
- Խնդիր 7: Տրված գրաֆի համար գտնել միջակայքերով գագաթային ներկում, երբ գագաթների վրա կան սահմանափակումներ:

Դիտարկված են գրաֆների հետևյալ դասերը. ծառեր, ցիկլերով ծառեր (երբ ամեն գագաթ պատկանում է առավելագույնը մեկ ցիկլի), լարերով ցիկլեր (սահմանում 2.2.1), երկկողմանի կակտուս գրաֆներ, կակտուս գրաֆներ, լրիվ երկկողմանի գրաֆներ $K_{n,m}$, լրիվ գրաֆներ K_n, բլոկ գրաֆներ: Աղյուսակ 1-ում ամփոփված են նախկինում ստացված արդյունքները և աշխատանքում ստացված արդյունքները: Խնդիր 1-ը ավելի հեշտ է, քան Խնդիր 2-ը, Խնդիր 2-ը և Խնդիր 3-ը ավելի հեշտ են, քան Խնդիր 4-ը, Խնդիր 4-ը ավելի հեշտ է, քան Խնդիր 5-ը, Խնդիր 6-ը ավելի հեշտ է, քան Խնդիր 7-ը: Հետևաբար՝ եթե ավելի դժվար խնդիրը լուծվում է բազմանդամային ժամանակում, ապա ավելի հեշտ խնդիրը նույնպես կարելի է լուծել բազմանդամային ժամանակում: Նման կերպով, եթե ավելի հեշտ խնդիրը NP-լրիվ է, ապա ավելի դժվար խնդիրը նույնպես NP-լրիվ է, քանի որ բոլոր դիտարկված խնդիրները NP դասից են:

Խնդիր 5-ի համար տրված են բազմանդամային ժամանակում աշխատող ալգորիթմներ ծառերի, ցիկլերով ծառերի և լարերով ցիկլերի դեպքում, ինչը նաև լուծում է Խնդիրներ 1, 2, 3, 4-ը այդ դասերի համար:

Խնդիր 7-ի համար տրված է բազմանդամային ժամանակում աշխատող ալգորիթմ կակտուս գրաֆների համար, ինչը նաև լուծում է Խնդիր 6-ը կակտուս գրաֆների համար, և Խնդիր 7-ը՝ երկկողմանի կակտուս գրաֆների, ցիկլերով ծառերի և ծառերի համար: Ցույց է տրված, որ Խնդիր 4-ը և Խնդիր 5-ը NP-լրիվ են երկկողմանի կակտուս գրաֆների համար, ինչը տալիս է որոշակի պատկերացում գրաֆների դասի սահմանի մասին, որտեղ բազմանդամային լուծելին դառնում է NP-լրիվ: Ապացույցները ուժի մեջ են նաև ճիշտ կողային ներկումների համար, ինչը նշանակում է, որ ճիշտ կողային

ներկում գտնելու խնդիրը գագաթների կամ կողերի վրա տրված սահմանափակումների դեպքում NP-լրիվ է կակտուս գրաֆերի և երկկողմանի կակտուս գրաֆերի համար: Այդ արդյունքները ապացուցված են 3-ցիկլ (երբ բոլոր ցիկլերը ունեն 3 երկարություն) և 4-ցիկլ կակտուս գրաֆների համար: Հետևաբար, կակտուս գրաֆների դեպքում անցումը սպեկտրների վրա տրված սահմանափակումներից դեպի գագաթների վրա տրված սահմանափակումներ դարձնում է խնդիրը NP-լրիվ:

Սա նշանակում է, որ կակտուս գրաֆների դեպքում, երբ անցում ենք կատարում սպեկտրների վրա տրված սահմանափակումներից դեպի գագաթների վրա տրված սահմանափակումներ, խնդիրը դառնում է NP-լրիվ:

Ապացուցված է, որ լրիվ և լրիվ երկկողմանի գրաֆերի համար Խնդիր 5-ը NP-լրիվ է: Ապացույցները ուժի մեջ են նաև ճիշտ կողային ներկումների դեպքում: Ապացուցված է, որ լրիվ երկկողմանի գրաֆների համար Խնդիր 6-ը NP-լրիվ է: Լրիվ երկկողմանի $K_{n,m}$ գրաֆների դեպքում տրված է Խնդիր 4-ը լուծող բազմանդամային ալգորիթմ, երբ $\sigma(n,m) = 1$:

Ցույց է տրված, որ զույգ բլոկների գրաֆների դեպքում (երբ ամեն բլոկ ունի զույգ քանակի գագաթներ) $w(G) \leq 2 \cdot (\Delta(G) - 1)$ և կառուցված է օրինակ, որի դեպքում $w(G) = 2 \cdot (\Delta(G) - 1)$, ինչը նշանակում է՝ այդ անհավասարությունը հնարավոր չի լավացնել զույգ բլոկների գրաֆների դասի համար: Ցույց է տրված, որ Խնդիր 5-ը և Խնդիր 4-ը NP-լրիվ են բլոկների գրաֆի համար: Խնդիր 6-ի համար տրված է բազմանդամային ժամանակում աշխատող ալգորիթմ բլոկների գրաֆների դեպքում: Նաև ցույց է տրված, որ Խնդիր 3-ը բլոկների գրաֆների համար կարելի է բերել Խնդիր 3-ին լրիվ գրաֆների համար:

Որոշ դեպքերում (օրինակ՝ Խնդիր 1-ը կակտուս գրաֆների համար, Խնդիր 3-ը լրիվ գրաֆների համար) չի հաջողվել ոչ գտնել խնդրի լուծման բազմանդամային ալգորիթմ, ոչ էլ ցույց տալ նրա NP-լրիվությունը:

Резюме

Раскраски графов с ограничениями имеют множество приложений, и изучение таких проблем для различных классов графов может быть полезно как для этих приложений, так и иметь определенное теоретическое значение.

В работе рассмотрены семь задач для некоторых классов графов.

- Задача 1: Нахождение какой-либо интервальной реберной раскраски заданного графа.
- Задача 2: Нахождение интервальной реберной раскраски с минимальным количеством цветов для заданного графа.
- Задача 3: Нахождение интервальной реберной раскраски при заранее заданных ограничениях на спектры вершин заданного графа.
- Задача 4: Нахождение интервальной реберной раскраски при заданных ограничениях на вершины заданного графа.
- Задача 5: Нахождение интервальной реберной раскраски при заданных ограничениях на ребра заданного графа.
- Задача 6: Нахождение раскраски вершин (предписанная раскраска) при заданных ограничениях на вершины заданного графа.
- Задача 7: Нахождение интервальной раскраски вершин при заданных ограничениях на вершины заданного графа.

Рассматриваемые классы графов: деревья, циклические деревья (каждая вершина принадлежит не более чем одному циклу), циклы с хордами (определение 2.2.1), двудольные кактусы, кактусы, полные двудольные графы $K_{n,m}$, полные графы K_n, блоковые графы. Таблица 1 представляет то, что было сделано ранее, и то, что сделано в данной работе. Задача 1 проще, чем Задача 2, Задача 2 и Задача 3 проще, чем Задача 4, Задача 4 проще, чем Задача 5, Задача 6 проще, чем Задача 7. Следовательно, если мы решим более сложную задачу за полиномиальное время, то мы также решим более простую задачу за полиномиальное время. Аналогично, если мы докажем, что более легкая задача является NP-полной, то более сложная задача тоже будет NP-полной, так как все рассматриваемые задачи из класса NP.

Предложены полиномиальные алгоритмы решения задачи 5 для деревьев, циклических деревьев и циклов с хордами, которые решают также задачи 1, 2, 3, 4 для этих же классов графов.

Предложен полиномиальный алгоритм решения задачи 7 для кактусов, который решает также задачи 6 и 7 для двудольных кактусов, циклических деревьев и деревьев. Доказано, что задачи 4 и 5 являются NP-полными для двудольных кактусов, что дает некоторое представление о той границе, которая разделяет в исследуемой тематике полиномиально разрешимые и NP-полные задачи. Доказательства остаются в силе также для задачи раскраски ребер, что означает, что нахождение раскраски ребер с заданными ограничениями на ребра или вершины является NP-полным для кактусов и двудольных кактусов. Эти же результаты доказаны для 3-цикловых (когда все циклы имеют длину

3) и 4-цикловых кактусов. Предложен полиномиальный алгоритм решения задачи 3 для кактусов. Это означает, что переход от ограничений на спектры вершин к ограничениям на вершины делает задачу NP-полной для кактусов.

Доказано, что задача 5 NP-полна для полных и полных двудольных графов. Доказательства остаются в силе и для рёберной раскраски, что означает, что проблема о предписанной раскраске рёбер является NP-полной для полных и полных двудольных графов. Доказано, что задача 6 также является NP-полной для полных двудольных графов. Предложен полиномиальный алгоритм решения задачи 4 в случае $K_{n,m}$ с $\sigma(n,m)=1$.

Показано, что для четных блоковых графов (каждый блок имеет четное число вершин) $w(G)\leq 2\cdot(\Delta(G)-1)$, и построен пример, для которого $w(G)=2\cdot(\Delta(G)-1)$, что означает, что неравенство $w(G)\leq 2\cdot(\Delta(G)-1)$ для класса четных блоковых графов улучшить невозможно. Доказано, что задачи 4 и 5 являются NP-полными для блоковых графов. В случае блоковых графов предложен полиномиальный алгоритм решения задачи 6. Показано, что задача 3 для блоковых графов может быть сведена к задаче 3 для полных графов.

В некоторых случаях (например задача 1 для кактусов, задача 3 для полных графов) не удалось найти ни полиномиального алгоритма решения, ни доказательства NP-полноты задачи.